Bayesian Networks

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Bayesian Networks

- Encoding of Conditional Independence Information in Directed Acyclic Graphs (DAG)
- Set of conditional probabilities
- Method for applying conditional probabilities to determine answer to (almost any) query
Markovian Parents

- $V$ is an ordered set of variables $X_1, X_2, \ldots X_n$.
- $P(V)$ is a joint probability distribution over $V$.
- Define the set of Markovian Parents of variable $X_j$, $PA_j$ as:
  - Minimal set of predecessors of $X_j$ such that $P(X_j|X_1, \ldots X_{j-1}) = P(X_j|PA_j)$
- The Markovian Parents of a variable $X_j$ are often (but not always) the direct causes of $X_j$.
For any set of variables $X_1, \ldots X_n$, we can calculate any row of the joint:

- $P(x_1, \ldots x_n) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2) \ldots P(x_n|x_1, x_2, \ldots x_{n-1})$

Using Markovian parents

- $P(x_1, \ldots x_n) = P(x_1)P(x_2|PA_2)P(x_3|PA_3) \ldots P(x_n|PA_n)$
We can create a DAG which represents conditional independence information using Markovian parents.

- Each variable is a node in the graph.
- For each variable $X_j$, add a directed link from all elements in $PA_j$ to $X_j$. 


Given a DAG of Markovian Parents, we know that every variable $X_i$ is independent of its ancestors, given its parents.

We also know quite a bit more.
To determine if a variable $X$ is conditionally independent of $Y$ given a set of variables $Z$:

- Examine all paths between $X$ and $Y$ in the graph
- Each node along a path can be “open” or “blocked”
  - A node at a head-to-tail or tail-to-tail junction is open if the node is not in $Z$, and closed otherwise.
  - A node at a head-to-head junction is open if the node or any of its descendants is not in $Z$, and closed otherwise.

Examples (on board)
To build a Bayesian Network:

- Select variables
- Order variables
  - Normally want a causal ordering
- Compute Markovian parents for each variable
- Compute $P(X_i|PA_i)$ for each variable
### Disease

<table>
<thead>
<tr>
<th>P(D)</th>
<th>D = ~d</th>
<th>D = d</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.999</td>
<td>0.001</td>
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</table>

### Test

<table>
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<tr>
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### Courier

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<td>T = t</td>
<td>0.1</td>
<td>0.9</td>
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Once we have our Bayesian Network, we will calculate probabilities using message passing.

Example:
- Leader of a group of troops wants to know how many soldiers are in the group.
- Sends a “count” message down line of soldiers.
- Gets a count reply back.
Platoon leader counting soldiers
Platoon leader counting soldiers, from middle of line
Platoon leader counting soldiers, with self-generating count signal
02-12: Message Passing

Leaderless Counting
A patient receives a “positive” result from the courier. Does the patient have the disease?

What is $P(d|c)$?

In general, what is $P(d|e)$, where $e$ is all the evidence that we have?
Break evidence $e$ into two pieces
- “causal evidence” or “causal support”, $e^+$
- “diagnostic evidence” or “evidential support” $e^-$

$$P(d|e^+_d, e^-_d) = \frac{P(d|e^+_d)P(e^-_d|d, e^+_d)}{P(e^-_d)}$$

$$= \frac{P(d|e^+_d)P(e^-_d|d)}{P(e^-_d)}$$

$$= \alpha P(d|e^+_d)P(e^-_d|d)$$
02-15: Renaming

\[ P(d|e_d^+, e_d^-) = \frac{P(d|e_d^+)P(e_d^-|d, e_d^+)}{P(e_d^-)} \]
\[ = \frac{P(d|e_d^+)P(e_d^-|d)}{P(e_d^-)} \]
\[ = \alpha P(d|e_d^+)P(e_d^-|d) \]

- \( \pi(x) = P(x|e_x^+) \)
- \( \lambda(x) = P(e_x^-|x) \)

Thus, \( P(d|e) = \alpha \pi(d) \lambda(d) \)
02-16: **Renaming**

\[ P(x|e_x^+, e_x^-) = \alpha P(x|e_x^+)P(e_x^-|x) \]
\[ = \alpha \pi(x)\lambda(x) \]

- \( \pi(x) \) is the “message” from upstream.
- \( \lambda(x) \) is the “message” from downstream.
Calculating $\pi(d)$

- $\pi(d)$ is the probability that $D = d$, given upstream evidence for $D$
- All we have for upstream evidence is the prior probability for $D$
- $\pi(d) = \text{Prior Probability on } d = P(d)$!
Calculating $\lambda(d)$

$$
\lambda(d) = P(e_d^- | d) \\
= \sum_{t \in T} P(e_d^- | d, t) P(t | d) \\
= \sum_{t \in T} P(e_t^- | t) P(t | d) \\
= \sum_{t \in T} \lambda(t) P(t | d)
$$
02-19: Calculating $\lambda(d)$

\[
\lambda(d) = \sum_{t \in T} \lambda(t) P(t|d)
\]

\[
\begin{align*}
\lambda(\neg d) &= \lambda(\neg t) P(\neg t|\neg d) + \lambda(t) P(t|\neg d) \\
\lambda(d) &= \lambda(\neg t) P(\neg t|d) + \lambda(t) P(t|d)
\end{align*}
\]
Calculating $\lambda(d)$

$$\lambda(d) = \sum_{t \in T} \lambda(t)P(t|d)$$

$$\lambda(D) = [\lambda(\neg d), \lambda(d)]$$
Calculating $\lambda(D)$:

$$\lambda(d) = \sum_{t \in T} \lambda(t)P(t|d)$$

$$\lambda(D) = [\lambda(\neg d), \lambda(d)]$$

$$= [\lambda(\neg t)P(\neg t|\neg d) + \lambda(t)P(t|\neg d), \lambda(\neg t)P(\neg t|d) + \lambda(t)P(t|d)]$$
Calculating $\lambda(D)$

$$\lambda(d) = \sum_{t \in T} \lambda(t)P(t|d)$$

$$\lambda(D) = [\lambda(\neg d), \lambda(d)]$$

$$= [\lambda(\neg t)P(\neg t|\neg d) + \lambda(t)P(t|\neg d), \lambda(\neg t)P(\neg t|d) + \lambda(t)P(t|d)]$$

$$= \begin{bmatrix} P(\neg t|\neg d) & P(t|\neg d) \\ P(\neg t|d) & P(t|d) \end{bmatrix} \begin{bmatrix} \lambda(\neg t) \\ \lambda(t) \end{bmatrix}$$
02-23: Calculating $\lambda(D)$

\[
\lambda(d) = \sum_{t \in T} \lambda(t) P(t|d)
\]

\[
\lambda(D) = [\lambda(\neg d), \lambda(d)]
\]
\[
= [\lambda(\neg t)P(\neg t|\neg d) + \lambda(t)P(t|\neg d), \lambda(\neg t)P(\neg t|d) + \lambda(t)P(t|d)]
\]
\[
= \begin{bmatrix}
P(\neg t|\neg d) & P(t|\neg d) \\
P(\neg t|d) & P(t|d)
\end{bmatrix}
\begin{bmatrix}
\lambda(\neg t) \\
\lambda(t)
\end{bmatrix}
\]
\[
= P(T|D)\lambda(T)
\]
\[
= M_{T|D}\lambda(T)
\]
Calculating $\lambda(D)$

- $\lambda(D) = M_{T|D} \lambda(T)$
- $\lambda(T) = M_{C|T} \lambda(C)$
- $\lambda(C) = ?$
  - What is the evidence that $C = \neg c, C = c$?
  - We know that $C = c$
  - $\lambda(C) = [0, 1]$
### Disease

<table>
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<th>( P(D) )</th>
<th>( D = \neg d )</th>
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<td>0.999</td>
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### Test

| \( P(T|D) \) | \( T = \neg t \) | \( T = t \) |
|-----------|---------------|---------------|
| \( D = \neg d \) | 0.9           | 0.1           |
| \( D = d \)   | 0.1           | 0.9           |

\( \lambda(C) = [0, 1] \)
02-26: Test / Courier Example

\begin{align*}
\begin{array}{c|cc}
\text{P}(D) & D = \sim d & D = d \\
\hline
0.999 & 0.001 \\
\end{array}
\end{align*}

\begin{align*}
\lambda(T) &= [0.05, 0.9] \\
\lambda(C) &= [0, 1]
\end{align*}

\begin{align*}
\begin{array}{c|cc}
P(T|D) & T = \sim t & T = t \\
\hline
D = \sim d & 0.9 & 0.1 \\
D = d & 0.1 & 0.9 \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c|cc}
P(C|T) & C = \sim c & C = c \\
\hline
T = \sim t & 0.95 & 0.05 \\
T = t & 0.1 & 0.9 \\
\end{array}
\end{align*}
02-27: Test / Courier Example

\[ P(D) \begin{array}{c|cc} D = \sim d & D = d \\ \hline 0.999 & 0.001 \end{array} \]

\[ \lambda(D) = [0.135, 0.815] \]

Disease

\[ \begin{array}{c|cc} P(T|D) & T = \sim t & T = t \\ \hline D = \sim d & 0.9 & 0.1 \\ D = d & 0.1 & 0.9 \end{array} \]

\[ \lambda(T) = [0.05, 0.9] \]

Test

\[ \begin{array}{c|cc} P(C|T) & C = \sim c & C = c \\ \hline T = \sim t & 0.95 & 0.05 \\ T = t & 0.1 & 0.9 \end{array} \]

\[ \lambda(C) = [0, 1] \]

Courier
Calculating $P(D|e)$

- $\lambda(C) = [0, 1]$
- $\lambda(T) = M_{C|T}\lambda(C) = [0.05, 0.9]$
- $\lambda(D) = M_{T|D}\lambda(T) = [0.135, 0.815]$

From before, $\pi(D) = P(D) = [0.999, 0.001]$

- $P(D|e) = \alpha \pi(D) \lambda(D)$
- $P(D|e) = \alpha[0.999, 0.001][0.135, 0.815]$
- $P(D|e) = \alpha[0.134865, 0.000815]$
  - $\alpha = 1/0.13568$
- $P(D|e) = [0.993993, 0.006007]$

02-28: Calculating $P(D|e)$
Calculating $P(T|e)$

- What if we wanted to calculate the probability that the test actually was positive, given that the courier delivered a positive result?

- $P(T|e) = \alpha \pi(T) \lambda(T)$

- We know $\lambda(T)$ from before

- What is $\pi(T)$?
02-30: Calculating $\pi(t)$

\[
\pi(t) = P(t|e_t^+)
\]

\[
= \sum_{d \in D} P(t|d, e_t^+) P(d|e_t^+)
\]

\[
= \sum_{d \in D} P(t|d, e_d^+) P(d|e_d^+)
\]

\[
= \sum_{d \in D} P(t|d) P(d|e_d^+)
\]

\[
= \sum_{d \in D} P(t|d) \pi(d)
\]
Calculating $\pi(t)$

\[
\pi(t) = P(t|e_t^+) \\
= \sum_{d \in D} P(t|d, e_t^+) P(d|e_t^+) \\
= \sum_{d \in D} P(t|d, e_d^+) P(d|e_d^+) \\
= \sum_{d \in D} P(t|d) P(d|e_d^+) \\
= \sum_{d \in D} P(t|d) \pi(d)
\]

\[
\pi(\neg t) = P(\neg t|\neg d) P(\neg d|e_d^+) + P(\neg t|d) P(d|e_d^+) \\
\pi(t) = P(t|\neg d) P(\neg d|e_d^+) + P(t|d) P(d|e_d^+)
\]
02-32: **Calculating** $\pi(T)$

$$
\pi(t) = \sum_{d \in D} P(t|d)\pi(d)
$$

$$
\pi(T) = [\pi(\neg t), \pi(t)] = [P(\neg t|\neg d)\pi(\neg d) + P(\neg t|d)\pi(d), P(t|\neg d)\pi(\neg d) + P(t|d)\pi(d)]
$$

$$
= [\pi(\neg d), \pi(d)]
$$

$$
\begin{bmatrix}
P(\neg t|\neg d) & P(t|\neg d) \\
P(\neg t|d) & P(t|d)
\end{bmatrix}
$$
Calculating $\pi(T)$

$\pi(D) = [0.999, 0.001]$  
$\lambda(D) = [0.135, 0.815]$  
$\pi(T) = [0.8992, 0.1008]$  
$\lambda(T) = [0.05, 0.9]$  
$\lambda(C) = [0, 1]$  

$P(D) | D = \sim d \quad D = d$  
$| 0.999 \quad 0.001$  

$P(T | D) | T = \sim t \quad T = t$  
$D = \sim d \quad 0.9 \quad 0.1$  
$D = d \quad 0.1 \quad 0.9$  

$P(C | T) | C = \sim c \quad C = c$  
$T = \sim t \quad 0.95 \quad 0.05$  
$T = t \quad 0.1 \quad 0.9$
02-34: Calculating \( BEL(T) = P(T|e) \)

- \( BEL(T) = \alpha \pi(T) \lambda(T) \)
  - \( \lambda(T) = [0.05, 0.9] \)
  - \( \pi(T) = [0.8992, 0.1008] \)
  - \( \pi(T) \lambda(T) = [0.04496, 0.09072] \)
  - \( \alpha = 1/(0.04496 + 0.09072) = 1/(0.13568) \)
- \( BEL(T) = [0.331368, 0.668632] \)
Calculating $\pi$ messages:
- $\pi(\text{root}) = \text{Prior on root}$
- For any other variable $X$ with parent $P$, $\pi(X) = \pi(P)M_{X|P}$

Calculating $\lambda$ messages:
- $\lambda(\text{leaf}) = \text{evidence for leaf}$
  - $([1, 1, \ldots, 1]$ if no evidence$)$
- For any other variable $X$ with child $C$, $\lambda(X) = M_{C|X}\lambda(C)$
Computation for Chains

- Send $\pi$ messages down
- Send $\lambda$ messages up
- For any variable $X$, we can calculate $BEL(X) = P(X|e)$ by multiplying the messages together, and normalizing
  - $P(X|e) = \alpha \lambda(X) \pi(X)$
    - (Pairwise multiplication)
Of course, variables can have > 2 values

Each variable can have a different number of values

Disease Example

- Doctor test for a disease
- Test can be positive, indeterminate, or negative
- Doctor discusses the result with the courier
- Courier delivers result
### Disease

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### Test

| $P(T|D)$ | $T = \text{neg}$ | $T = \text{ind}$ | $T = \text{pos}$ |
|----------|------------------|------------------|------------------|
| $D = \neg d$ | 0.8              | 0.1              | 0.1              |
| $D = d$    | 0.1              | 0.1              | 0.8              |

### Courier

| $P(C|T)$ | $C = \neg c$ | $C = c$ |
|----------|--------------|---------|
| $T = \text{neg}$ | 0.9          | 0.1     |
| $T = \text{ind}$  | 0.5          | 0.5     |
| $T = \text{pos}$  | 0.1          | 0.9     |
02-39: Variable # of Values / Variables

\[
\begin{array}{c|cc}
\text{P(D)} & D = \sim d & D = d \\
\hline
0.999 & 0.001
\end{array}
\]

\[\lambda(D) = [0.22, 0.78]\]

\[\lambda(T) = [0.1, 0.5, 0.9]\]

\[\lambda(C) = [0, 1]\]
What if some of the nodes have $>1$ child?

Example: Send message via two different couriers
**02-41: Computation for Trees**

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Disease

| P(T|D) | T = ~t | T = t |
|------|--------|-------|
| D = ~d | 0.9   | 0.1   |
| D = d   | 0.1   | 0.9   |

Test

| P(C1|T) | C = ~c | C = c |
|------|--------|-------|
| T = ~t | 0.95   | 0.05  |
| T = t   | 0.1    | 0.9   |

| P(C2|T) | C = ~c | C = c |
|------|--------|-------|
| T = ~t | 0.95   | 0.05  |
| T = t   | 0.1    | 0.9   |
02-42: Computation for Trees

- How do we send $\lambda$ messages in trees?
- Courier example: What is $\lambda(T)$, which is the probability of the downstream evidence given the test, if both couriers give a positive response?
- We will need to combine the messages that we get from each child into a single $\lambda$ message
  - Use this $\lambda$ message to compute $BEL(T)$
  - Use this $\lambda$ message to send a message to $D$
Calculating $\lambda(t)$

$$
\lambda(t) = P(e_t^- | t) \\
= P(e_{C1}^-, e_{C2}^- | t) \\
= P(e_{C1}^- | t)P(e_{C2}^- | t) \\
= \sum_{c_1 \in C1} P(e_{C1}^- | c_1, t)P(c_1 | t) \sum_{c_2 \in C2} P(e_{C2}^- | c_2, t)P(c_2 | t) \\
= \sum_{c_1 \in C1} P(e_{C1}^- | c_1)P(c_1 | t) \sum_{c_2 \in C2} P(e_{C2}^- | c_2)P(c_2 | t) \\
= \sum_{c_1 \in C1} \lambda(c_1)P(c_1 | t) \sum_{c_2 \in C2} \lambda(c_2)P(c_2 | t)
$$
02-44: Calculating $\lambda(T)$

\[
\lambda(t) = \sum_{c_1 \in C_1} \lambda(c_1) P(c_1|t) \sum_{c_2 \in C_2} \lambda(c_2) P(c_2|t)
\]

\[
\lambda(T) = M_{C_1|T} \lambda(C_1) \ast M_{C_2|T} \lambda(C_2) = \lambda_{C_1}(T) \ast \lambda_{C_2}(T)
\]
02-45: Computation for Trees

\[
\begin{array}{c|cc}
P(D) & D = \sim d & D = d \\
\hline
0.999 & 0.001 \\
\end{array}
\]

\[\lambda(D) = [0.08325, 0.72925]\]

\[\lambda(T) = [0.0025, 0.81]\]

\[\lambda_{C1}(T) = [0.05, 0.9]\]

\[\lambda_{C2}(T) = [0.05, 0.9]\]

\[
\begin{array}{c|cc}
P(T|D) & T = \sim t & T = t \\
\hline
D = \sim d & 0.9 & 0.1 \\
D = d & 0.1 & 0.9 \\
\end{array}
\]

\[
\begin{array}{c|cc}
P(C1|T) & C = \sim c & C = c \\
\hline
T = \sim t & 0.95 & 0.05 \\
T = t & 0.1 & 0.9 \\
\end{array}
\]

\[\lambda(C1) = [0, 1]\]

\[
\begin{array}{c|cc}
P(C2|T) & C = \sim c & C = c \\
\hline
T = \sim t & 0.95 & 0.05 \\
T = t & 0.1 & 0.9 \\
\end{array}
\]

\[\lambda(C2) = [0, 1]\]
• $BEL(D) = \alpha \pi(D) \lambda(D)$
  • $\pi(D) = [0.999, 0.001]$  
  • $\lambda(D) = [0.08325, 0.72925]$  
  • $\pi(D)\lambda(D) = [0.0831667, 0.00072925]$  
  • $\alpha = 1/(0.08389595)$  
• $BEL(D) = [0.991308, 0.008692]$
Sending $\pi$ Messages in Trees

- $\pi(x) = P(x|e_x^+)$
- That is, $\pi(x)$ is $P(X = x)$, given all upstream evidence from $X$

\[ \pi(x) \]

- $\pi(X) = P(P|e_X^+)P(X|P)$
- $\pi(P) \ast \lambda_{\text{other children of } P} P(P)M_{X|P}$
- $(BEL(P)/\lambda_X(P))M_{X|P}$
  - Pairwise division
What is the probability that Courier 1 will give a positive result, given that Courier 2 gave a positive result?

$P(C1|e)$

Evidence $e$ is the prior probability for disease, and the fact that Courier 2 gave a positive result
What is the probability that Courier 1 will give a positive result, given that Courier 2 gave a positive result?

$P(C1|e)$

Evidence $e$ is the prior probability for disease, and the fact that Courier 2 gave a positive result

$\pi(C1) = \alpha \pi(T) \ast \lambda_{C2}(T) M_{C1|T}$
02-50: Computation for Trees

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</table>

\[ \pi(D) = \{0.999, 0.001\} \]

| P(T | D) | T = ~t | T = t |
|-----|------|-------|
|     | D = ~d | 0.9   | 0.1 |
|     | D = d   | 0.1   | 0.9 |

\[ \pi(T) = \{0.8992, 0.1008\} \]

\[ \pi(T) \lambda_{C2}(T) = \{0.04496, 0.09072\} \]

\[ \lambda_{C2}(T) = \{0.05, 0.9\} \]

| P(C1 | T) | C = ~c | C = c |
|-----|------|-------|
|     | T = ~t | 0.95  | 0.05 |
|     | T = t   | 0.1   | 0.9 |

\[ \pi(C1) = \alpha{0.051884, 0.083896} \]

\[ = \{0.382952, 0.619232\} \]
For root variable $R$, $\pi(R) = \text{Prior on } R$

For unobserved leaf variables $L$, $\lambda(L) = [1, 1, \ldots, 1]$

For leaf variables $L$ observed to have the value $l_k$, $\lambda(L) = [0, \ldots, 0, 1, 0, \ldots 0]$ – the $k^{th}$ element is 1, all others are 0

Pass $\pi$ and $\lambda$ messages through the tree
  - Multiply $\pi$ message by $\lambda$ messages from other children, then multiply the result by the link matrix
  - Multiply link matrix by $\lambda$ messages
    - Multiple Children – multiply $\lambda$ messages
Add a gender variable

Test for disease depends upon gender, as well as disease state

Need to expand link matrix for test to include gender
  • Need \( P(t|g, d) \) for all values of \( t, g, d \)
02-53: Multiple Parents (Polytrees)

\[
\begin{array}{c|cc}
D & \sim d & d \\
\hline
P(D) & 0.999 & 0.001
\end{array}
\]

\[
\begin{array}{c|cc}
G & m & f \\
\hline
P(G) & 0.5 & 0.5
\end{array}
\]

\[
\begin{array}{c|cc}
T & \sim t & t \\
\hline
P(T|D,G) & \sim d, m & 0.9 & 0.1 \\
& \sim d, f & 0.8 & 0.2 \\
& d, m & 0.1 & 0.9 \\
& d, f & 0.2 & 0.8
\end{array}
\]

\[
\begin{array}{c|cc}
C & \sim c & c \\
\hline
P(C|T) & \sim t & 0.95 & 0.05 \\
& t & 0.1 & 0.9
\end{array}
\]
Calculating $\pi()$ in Polytrees

- For each parent $X$, we have $P(X|e^+)$
  - $P(D) = [0.999, 0.001], P(G) = [0.5, 0, 5]$
- We need the probabilities for all combinations of parents
  - $P(\neg d, m), P(\neg d, f), P(d, m), P(d, f)$
- Parents are *independent* given upstream evidence
  - $P(\neg d, m) = P(\neg d)P(m)$
Calculating $\pi()$ in Polytrees

- We have $[P(\neg d), P(d)]$ and $[P(m), P(f)]$
- We need $[P(\neg d, m), P(\neg d, f), P(d, m), P(d, f)]$
  - $P(\neg d, m) = P(\neg d)P(m), P(\neg d, f) = P(\neg d)P(f)$, etc.
- $P(\neg d, m) = 0.999 \times 0.5$, $P(\neg d, f) = 0.999 \times 0.5$, $P(d, m) = 0.001 \times 0.5$, $P(d, f) = 0.001 \times 0.5$
- $P(D, G) = [0.4995, 0.4995, 0.0005, 0.0005]$
- $\pi(T) = \begin{bmatrix} P(\neg t|\neg d, m) & P(t|\neg d, m) \\ P(\neg t|\neg d, f) & P(t|\neg d, f) \\ P(\neg t|d, m) & P(t|d, m) \\ P(\neg t|d, f) & P(t|d, f) \end{bmatrix}$
02-56: Calculating \( \pi(T) \)

\[
\begin{bmatrix}
\pi(\neg d,m) & \pi(\neg d,f) & \pi(d,m) & \pi(d,f)
\end{bmatrix}
\begin{bmatrix}
P(\neg t|\neg d, m) & P(t|\neg d, m) \\
P(\neg t|\neg d, f) & P(t|\neg d, f) \\
P(\neg t|d, m) & P(t|d, m) \\
P(\neg t|d, f) & P(t|d, f)
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
0.4995 & 0.4995 & 0.0005 & 0.0005
\end{bmatrix}
\begin{bmatrix}
0.9 & 0.1 \\
0.8 & 0.2 \\
0.1 & 0.9 \\
0.2 & 0.8
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0.8493 & 0.1507
\end{bmatrix}
\]
What is our belief that the test actually is positive, given that the courier delivers a positive message?

- $\pi(T) = [0.8493, 0.1507]
- $\lambda(T) = \begin{bmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\lambda(T) = [0.05, 0.9]
- BEL(T) = \alpha[0.42465, 0.13565] \ (\alpha = 1/0.5603)$
- $BEL(T) = [0.757898, 0.242102]$
Calculating $\pi()$ in Polytrees

- To calculate $\pi(X)$, when $X$ has multiple parents $m$:
  - For each parent $Y_k$ of $X$, calculate $P(Y_k|e_X^+)$
    (Define message from $Y_k$ to $X$, $\pi_X(Y_k) = (Y_k|e_X^+)$)
    - If $X$ is the only child of $Y_k$, $\pi_X(Y_k) = \pi(Y_k)$
    - If $Y_k$ has children $C_1 \ldots C_j$ other than $X$, then
      $\pi_X(Y_k) = \pi(Y_k) \prod_{i=i\ldots j} \lambda_{C_i}(Y)$
      (That is, $\pi_X(Y_k) = \text{BEL}(Y)/\lambda_X(Y)$)
  - Combine the $\pi_X$ messages from all the parents, and multiply the result by the link matrix
    $M_{X|Y_1\ldots Y_m}$ to get $\pi(X)$
02-59: **Calculating** $\lambda()$ in Polytrees

<table>
<thead>
<tr>
<th>$P(D)$</th>
<th>$D = \sim d$</th>
<th>$D = d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(D)$</td>
<td>0.999</td>
<td>0.001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$P(G)$</th>
<th>$G = m$</th>
<th>$G = f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(G)$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

| $P(T|D,G)$ | $T = \sim t$ | $T = t$ |
|------------|--------------|--------|
| $\sim d, m$ | 0.9         | 0.1    |
| $\sim d, f$ | 0.8         | 0.2    |
| $d, m$     | 0.1         | 0.9    |
| $d, f$     | 0.2         | 0.8    |

| $P(C|T)$ | $C = \sim c$ | $C = c$ |
|----------|--------------|--------|
| $T = \sim t$ | 0.95      | 0.05   |
| $T = t$   | 0.1         | 0.9    |

**Disease**

**Gender**

**Test**

**Courier**
How do we send a $\lambda$ message up to Disease, given the combined link matrix for Disease and Gender?

\[
\begin{bmatrix}
P(\neg t|\neg d, m) & P(t|\neg d, m) \\
P(\neg t|\neg d, f) & P(t|\neg d, f) \\
P(\neg t|d, m) & P(t|d, m) \\
P(\neg t|d, f) & P(t|d, f)
\end{bmatrix}
\]

If we knew that the gender was definitely male, then we could select the appropriate two rows, to create a 2x2 matrix:

\[
\begin{bmatrix}
P(\neg t|\neg d, m) & P(t|\neg d, m) \\
P(\neg t|d, m) & P(t|d, m)
\end{bmatrix}
\]
How do we send a $\lambda$ message up to Disease, given the combined link matrix for Disease and Gender?

$$\begin{bmatrix}
P(\neg t|\neg d, m) & P(t|\neg d, m) \\
P(\neg t|\neg d, f) & P(t|\neg d, f) \\
P(\neg t|d, m) & P(t|d, m) \\
P(\neg t|d, f) & P(t|d, f)
\end{bmatrix}$$

If we knew that the gender was definitely female, then we could select the appropriate two rows, to create a 2x2 matrix:

$$\begin{bmatrix}
P(\neg t|\neg d, f) & P(t|\neg d, f) \\
P(\neg t|d, f) & P(t|d, f)
\end{bmatrix}$$
If we knew the value of Gender, we could select the correct rows to build the appropriate link matrix to send the lambda message.

We don’t know for certain the value of Gender, but we do know the probability $G_j$, given evidence upstream of $T$:

- $P(G|e_T^+) = \pi_T(G) = \pi(G) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$

We can then average the rows:

$$
\begin{bmatrix}
P(\neg t | \neg d, m) P(m) + P(\neg t | \neg d, f) P(f) & P(t | \neg d, m) P(m) + P(t | \neg d, f) P(f) \\
P(\neg t | d, m) P(m) + P(\neg t | d, f) P(f) & P(t | d, m) P(m) + P(t | d, f) P(f)
\end{bmatrix}
$$
## Calculating \( \lambda() \) in Polytrees

### Original Link Matrix \( M_{T|D,C} \)

| \( P(T|D,C) \) | \( T = \neg t \) | \( T = t \) |
|-----------------|-----------------|-----------------|
| \( \neg d, m \) | 0.9             | 0.1             |
| \( \neg d, f \) | 0.8             | 0.2             |
| \( d, m \)      | 0.1             | 0.9             |
| \( d, f \)      | 0.2             | 0.8             |

### Revised Link Matrix \( M_{T|D} \)

| \( P(T|D) \)   | \( T = \neg t \) | \( T = t \) |
|----------------|-----------------|-----------------|
| \( \neg d \)   | 0.85            | 0.15            |
| \( d \)        | 0.15            | 0.85            |
02-64: Calculating \( \text{BEL}(D) \)

\[
\lambda(D) = \begin{bmatrix} 0.85 & 0.15 \\ 0.15 & 0.85 \end{bmatrix} \begin{bmatrix} 0.05 \\ 0.9 \end{bmatrix} \\
= \begin{bmatrix} 0.1775 & 0.7725 \end{bmatrix}
\]

\[
\pi(D) = \begin{bmatrix} 0.999 & 0.001 \end{bmatrix}
\]

\[
\text{BEL}(D) = \alpha \pi(D) \lambda(D)
\]

\[
= \alpha \begin{bmatrix} 0.177323 & 0.0007725 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0.99566 & 0.00434 \end{bmatrix}
\]
02-65: **Complete Polytree Example**

<table>
<thead>
<tr>
<th>P(D)</th>
<th>D = ~d</th>
<th>D = d</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.999</td>
<td>0.001</td>
</tr>
</tbody>
</table>

| P(T|D,G) | T = ~t | T = t |
|--------|--------|-------|
| ~d, m  | 0.9    | 0.1   |
| ~d, f  | 0.8    | 0.2   |
| d, m   | 0.1    | 0.9   |
| d, f   | 0.2    | 0.8   |

<table>
<thead>
<tr>
<th>P(G)</th>
<th>G = m</th>
<th>G = f</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

| P(N|G) | sue | chris | john |
|------|-----|-------|------|
| G = m| 0.1 | 0.4   | 0.5  |
| T = f| 0.5 | 0.4   | 0.1  |

| P(C|T) | C = ~c | C = c |
|------|--------|-------|
| T = ~t| 0.9    | 0.1   |
| T = t  | 0.1    | 0.9   |

(Courier link matrices the same)
Complete Polytree Example

- Find $BEL(D)$, given that:
  - Both couriers return a positive result
  - Patients name is John
Polytree Example: $\lambda$s

|       | $P(C|T)$ | $C = \sim c$ | $C = c$ |
|-------|----------|--------------|---------|
| $T = \sim t$ | 0.9      | 0.1          |
| $T = t$    | 0.1      | 0.9          |

- $\lambda(C_1) = \lambda(C_2) = [0, 1]$
- $\lambda_{C_1}(T) = [0.1, 0.9]$
- $\lambda_{C_2}(T) = [0.1, 0.9]$
- $\lambda(T) = [0.01, 0.81]$
Gender

| Gender | P(N|G) | Name   |
|--------|-------|--------|
| G = m  | 0.1   | sue    | 0.4    | chris  | 0.5    | john   |
| T = f  | 0.5   | 0.4    | 0.1    |        |        |        |

- $\lambda(N) = [0, 0, 1]$
- $\lambda(G) = [0.5, 0.1]$
02-69: Polytree Example: λs

<table>
<thead>
<tr>
<th>P(G)</th>
<th>G = m</th>
<th>G = f</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Gender

π_T(G) = α[0.25, 0.05]
= [0.8333, 0.1667]

λ(G) = [0.5, 0.1]
Polytree Example: λs

| P(T|D,G) | T = ~t | T = t |
|---------|--------|--------|
| ~d, m  | 0.9    | 0.1    |
| ~d, f  | 0.8    | 0.2    |
| d, m   | 0.1    | 0.9    |
| d, f   | 0.2    | 0.8    |

Disease

Gender

\[ \pi_T(G) = [0.8333, 0.1667] \]

\[ \lambda(T) = [0.01, 0.81] \]
### Polytree Example: λs

| P(T|D) | T = ~t | T = t |
|-------|--------|-------|
| ~d    | 0.8833 | 0.1167|
| d     | 0.1167 | 0.8833|

λ(T) = [0.01, 0.81]

π_T(G) = [0.8333, 0.1667]

Disease \(\rightarrow\) Test \(\rightarrow\) Gender
02-72: Polytree Example: \( \lambda \)s

\[ \lambda(D) = [0.1034, 0.7166] \]

\[ \pi_T(G) = [0.8333, 0.1667] \]

\[ \lambda(T) = [0.01, 0.81] \]

| \( P(T|D) \) | \( T = \sim t \) | \( T = t \) |
|-------------|----------------|----------------|
| ~d          | 0.8833         | 0.1167         |
| d           | 0.1167         | 0.8833         |
02-73: **Polytree Example: \( \lambda \)s**

\( \lambda(D) = [0.1034, 0.7166] \)

\[
\begin{array}{c|cc}
D & \sim t & t \\
\hline
\sim d & 0.8833 & 0.1167 \\
d & 0.1167 & 0.8833 \\
\end{array}
\]

\( \pi_T(G) = [0.8333, 0.1667] \)

\( \lambda(T) = [0.01, 0.81] \)

\[
\begin{align*}
BEL(D) &= \alpha \pi(D) \lambda(D) \\
BEL(D) &= \alpha [0.9999, 0.001] [0.1034, 0.7166] \\
BEL(D) &= \alpha [0.1033, 0.0007] \\
BEL(D) &= \alpha [0.9933, 0.0067]
\end{align*}
\]
What if we observe a variable that is not a leaf? 
  • For instance, we observe the actual test result 
  • Add a “phantom child” 
  • Set $\lambda$ message from that child to $[0, \ldots, 0, 1, 0, \ldots, 0]$, where the 1 occurs at the observed value 
  • This $\lambda$ message will override all other evidence for the node